

SEIDEL ELEMENTS AND MIRROR TRANSFORMATIONS

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ABSTRACT. The goal of this article is to give a precise relation between the mirror symmetry transformation of Givental and the Seidel elements for a smooth projective toric variety X with $-K_X$ nef. We show that the Seidel elements entirely determine the mirror transformation and mirror coordinates.

January 15, 2013

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1. INTRODUCTION

Let X be a smooth projective toric variety. The variety X can be explicitly written as the symplectic reduction of the Hermitian space \mathbb{C}^m by a Hamiltonian action of a torus $(S^1)^r$, where r is the Picard number of X . Let D_1, \dots, D_m denote the classes in $H^2(X)$ Poincaré dual to the toric divisors. Let t_i denote the coordinates in $H^2(X)$ with respect to an integral, nef basis p_1, \dots, p_r , and let $q_i = \exp(t_i)$ be the exponential coordinates. Recall that the mirror theorem of Givental [9] states that if $c_1(X) = -K_X = D_1 + \dots + D_m$ is semipositive (nef),

E.G. is supported by NSF grant DMS-1104670 and H.I. is supported by Grant-in-Aid for Young Scientists (B) 22740042 .

then the cohomology valued function (of the B-model)

$$I_X(y, z) = e^{\sum_{i=1}^r p_i \log y_i / z} \sum_{d \in \text{NE}(X)_{\mathbb{Z}}} \prod_{i=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_i + kz)}{\prod_{k=-\infty}^{\langle D_i, d \rangle} (D_i + kz)} \right) y_1^{d_1} \cdots y_r^{d_r}$$

determines the J-function $J_X(q, z)$ (of the the A -model or Gromov-Witten theory) (1)

$$J_X(q, z) = e^{\sum_{i=1}^r p_i \log q_i / z} \left(1 + \sum_{\alpha} \sum_{d \in \text{NE}(X)_{\mathbb{Z}} \setminus \{0\}} \left\langle \frac{\phi_{\alpha}}{z(z - \psi)} \right\rangle_{0,1,d}^X \phi^{\alpha} q_1^{d_1} \cdots q_r^{d_r} \right)$$

via a change of coordinates $\log q_i = \log y_i + g_i(y)$, $i = 1, \dots, r$, in such a way that $I_X(y, z) = J_X(q, z)$. Here the variables y_1, \dots, y_r of the B-model are called *mirror coordinates* and this change of variables is called *mirror transformation* (or *mirror map*). This relation can be used to show that the small quantum cohomology ring $QH(X)$ differs from the original presentation suggested by Batyrev [2] only by this change of coordinates. We refer to Givental [9] and the text by Cox and Katz [4] for further details on this discussion.

Let (Y, ω) denote a symplectic manifold. For a loop λ in the group of Hamiltonian symplectomorphisms on Y , Seidel [19], assuming Y monotone, constructed an invertible element $S(\lambda)$ in quantum cohomology counting sections of the associated clutched Hamiltonian fibration $E_{\lambda} \rightarrow \mathbb{P}^1$ with fibre Y . The *Seidel element* $S(\lambda)$ defines an element in $\text{Aut}(QH(Y))$ via quantum multiplication, and the association $\lambda \mapsto S(\lambda)$ a representation of $\pi_1(\text{Ham}(Y))$ on $QH(Y)$. This construction was later extended for all symplectic manifolds, see for instance McDuff and Tolman [17] where Seidel's construction was used to study the underlying symplectic topology in toric manifolds.

In the case where the loop λ is a (relatively simple) circle action, the asymptotic form of $S(\lambda)$ can be written explicitly in terms of geometric and Morse theoretic information [17, Theorem 1.10]. Regarding X as a Hamiltonian space, they consider the Seidel element¹ \tilde{S}_j associated to an action λ_j that fixes the toric divisor D_j (see Section 3.2). It is proved that \tilde{S}_j is a series of the form $\tilde{S}_j = D_j + O(q)$ if $-K_X$ is nef, and $\tilde{S}_j = D_j$ in the Fano case ($-K_X$ is ample). Moreover, it is shown that these elements satisfy the following Batyrev's relations:

$$(2) \quad \prod_{j: \langle D_j, d \rangle > 0} \tilde{S}_j^{\langle D_j, d \rangle} = q^d \prod_{j: \langle D_j, d \rangle < 0} \tilde{S}_j^{-\langle D_j, d \rangle} \quad \text{in } QH(X)$$

where $d \in H_2(X, \mathbb{Z})$. This shows that $QH(X)$ is abstractly isomorphic to the Batyrev presentation [17, Proposition 5.2]. Based on this, one can conjecture that Seidel elements should be closely related or even equivalent to the mirror transformation. In a recent paper Fukaya, Oh, Ohta and Ono [7] have used Seidel elements in the mirror symmetry context as well.

We begin with a calculation of Seidel elements. It turns out that \tilde{S}_j appears as a coefficient of the J-function J_{E_j} of the associated bundle E_j (Proposition 2.5), and thus one can use the mirror transformation for E_j (which is toric as well) to calculate \tilde{S}_j . Let (m_{ij}) be the matrix of toric divisors such that $D_j = \sum_{i=1}^r m_{ij} p_i$.

¹Here \tilde{S}_j is a variant of the Seidel element $S_j = S(\lambda_j)$ given by $S_j = q_0 \tilde{S}_j$, where q_0 is the variable corresponding to the maximal section class of the associated bundle E_j .

As mirror analogues of Seidel elements, we introduce the *Batyrev elements* $\tilde{D}_j = D_j + O(y) \in H^*(X)[[y]]$ by

$$(3) \quad \tilde{D}_j = \sum_{i=1}^r m_{ij} \tilde{p}_i \quad \text{with} \quad \tilde{p}_i = \sum_{k=1}^r \frac{\partial \log q_k}{\partial \log y_i} p_k.$$

These elements satisfy Batyrev's relations (2) with q^d there replaced by y^d (Proposition 3.8). We can calculate them as explicit hypergeometric series in y (Lemma 3.17). Note that the Batyrev elements and the Seidel elements satisfy the *same* relation in *different* coordinates. We see that they only differ by a function multiple.

Theorem 1.1 (Theorem 3.13, Lemma 3.16). *Let X be a smooth projective toric variety with $-K_X$ nef. The Seidel element associated to the toric divisor D_j is given by*

$$\tilde{S}_j(q_1 \dots q_r) = \exp \left(-g_0^{(j)}(y_1, \dots, y_r) \right) \tilde{D}_j(y_1, \dots, y_r).$$

The correction term $g_0^{(j)}$ here is an explicit hypergeometric series (26) in y .

Next we ask the converse: whether the Seidel elements determine the mirror transformation. One can see from the definition (3) that the Batyrev elements $\tilde{D}_1, \dots, \tilde{D}_m$ can determine the Jacobian of the mirror map, in particular, the mirror map itself. Therefore, it is important to know when $g_0^{(j)}$ vanishes. We show that $g_0^{(j)}$ vanishes if and only if $-K_X$ is **big** on the toric divisor D_j , i.e. $(-K_X)^{n-1} \cdot D_j > 0$. In terms of the fan of X , this is also equivalent to the primitive generator b_j of the corresponding 1-cone being a vertex of the fan polytope (Proposition 4.3). Another important fact is that the Batyrev elements \tilde{D}_j satisfy the same linear relations as the toric divisors do:

$$(4) \quad \sum_{i=1}^m c_i \tilde{D}_i = 0 \quad \text{whenever} \quad \sum_{i=1}^m c_i D_i = 0.$$

The Seidel elements \tilde{S}_j do not necessarily satisfy the linear relations. The partial vanishing of $g_0^{(j)}$ and these linear relations are enough to reconstruct the Batyrev elements from the Seidel elements.

Theorem 1.2. *For a smooth projective toric variety X with $-K_X$ nef, the Seidel elements \tilde{S}_j entirely determine the Batyrev elements \tilde{D}_j and in particular the mirror transformation. More precisely the Batyrev elements $\tilde{D}_j \in H^*(X)[[q]]$, $j = 1, \dots, m$ are uniquely characterized by the following conditions:*

- (i) $\tilde{D}_j = H_j \tilde{S}_j$ for some $H_j \in \mathbb{Q}[[q]]$;
- (ii) $\tilde{D}_j = \tilde{S}_j$ if $(-K_X)^{n-1} \cdot D_j > 0$;
- (iii) \tilde{D}_j satisfy the linear relations (4).

We add a remark on mirror symmetry. The mirror coordinates y_1, \dots, y_r here represent the complex moduli of the mirror Landau-Ginzburg model. There are no preferred choices of coordinates on the complex moduli space and both q and y serve as local coordinates. However the y coordinates are considered to represent a global algebraic structure of the complex moduli space. Therefore our result suggests that Gromov-Witten theory itself (via torus action and Seidel elements) can reconstruct a global algebraization of the Kähler moduli space which is a priori a formal germ at $q = 0$.

Acknowledgments. We thank David Cox, Dusa McDuff, Kaoru Ono and Chris Woodward for useful discussions regarding an early draft of the paper. We also thank Changzheng Li and Kwokwai Chan for helpful comments on an earlier version of the paper. The authors are grateful for the hospitality at the Centre International de Rencontres Mathématiques in Luminy, where this project was started. E.G. is supported by NSF grant DMS-1104670 and H.I. is supported by Grant-in-Aid for Young Scientists (B) 22740042.

2. SEIDEL ELEMENTS AND J-FUNCTIONS

We introduce the main notation and constructions of Seidel element and explain its relation to the J-function.

2.1. Generalities. We begin with the definition of the Seidel element. Let X be a smooth projective variety, equipped with a \mathbb{C}^\times action. Each \mathbb{C}^\times -orbit in X contains a fixed point in its closure, and thus the associated S^1 -action on X is Hamiltonian by Frankel's theorem [6]. In this paper we will restrict to this case, however the construction works more generally in the symplectic category. The original definition is due to Seidel [19] for monotone symplectic manifolds. The reader can consult [16] for a detailed exposition and [12, 14] for the construction in general case. For relations with the computation of small quantum cohomology rings see [14, 10, 15, 17].

Definition 2.1. The **associated bundle** of the \mathbb{C}^\times -action is the X -bundle over \mathbb{P}^1

$$E := X \times (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^\times \rightarrow \mathbb{P}^1,$$

where \mathbb{C}^\times acts with the standard diagonal action $\lambda \cdot (x, z) = (\lambda x, \lambda z)$.

Let ϕ_1, \dots, ϕ_N denote a basis for the rational cohomology $H^*(X; \mathbb{Q})$. By abuse of notation, every time we omit coefficients we mean rational cohomology. Let ϕ^1, \dots, ϕ^N denote the dual basis, that is $(\phi_i, \phi^j) = \delta_{ij}$ where (\cdot, \cdot) is the usual pairing in cohomology. There is a (non-canonical) splitting [14]

$$H^*(E) \cong H^*(X) \otimes H^*(\mathbb{P}^1).$$

We let $\hat{\phi}_1, \dots, \hat{\phi}_M$ denote a basis for $H^*(E)$, and let $\hat{\phi}^1, \dots, \hat{\phi}^M$ denote the dual basis. There is a unique \mathbb{C}^\times -fixed component $F_{\max} \subset X^{\mathbb{C}^\times}$ such that the normal bundle of F_{\max} has only negative \mathbb{C}^\times -weights. When we take a Hamiltonian function for the S^1 -action, F_{\max} is the maximum set of the Hamiltonian. Each fixed point $x \in X^{\mathbb{C}^\times}$ defines a section σ_x of E . We denote by σ_0 the section associated to a fixed point in F_{\max} . This maximal section defines a splitting

$$(5) \quad H_2(E, \mathbb{Z}) / \text{tors} \cong \mathbb{Z}[\sigma_0] \oplus (H_2(X, \mathbb{Z}) / \text{tors})$$

Let $\text{NE}(X) \subset H_2(X, \mathbb{R})$ denote the Mori cone, that is the cone generated by effective curves and set $\text{NE}(X)_{\mathbb{Z}} := \text{NE}(X) \cap (H_2(X, \mathbb{Z}) / \text{tors})$. We introduce $\text{NE}(E)$ and $\text{NE}(E)_{\mathbb{Z}}$ similarly.

Lemma 2.2. $\text{NE}(E)_{\mathbb{Z}} = \mathbb{Z}_{\geq 0}[\sigma_0] + \text{NE}(X)_{\mathbb{Z}}$.

Proof. The associated bundle E has the $T^2 = \mathbb{C}^\times \times \mathbb{C}^\times$ action defined by $(t_1, t_2) \cdot [x, (z_1, z_2)] = [t_1 x, (z_1, t_2 z_2)]$. By the T^2 -action, every curve can be deformed to a sum of T^2 -invariant curves in the same homology class. A T^2 -invariant curve is either contained in the fibres at 0, ∞ or in the subspace $F \times \mathbb{P}^1 \subset E$ for some

fixed component $F \subset X^{\mathbb{C}^\times}$. Therefore it suffices to show that the section class $[\sigma_x]$ associated to a fixed point $x \in X^{\mathbb{C}^\times}$ is of the form $[\sigma_0] + d$ for some $d \in \text{NE}(X)_{\mathbb{Z}}$. Take a nontrivial \mathbb{C}^\times -orbit O in X and consider its closure $\overline{O} \cong \mathbb{P}^1$. This gives an embedding of the Hirzebruch surface $\mathbb{F}_k \cong \overline{O} \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$ into E , where k is the order of the stabilizer at O . It is well known that the maximal section class in \mathbb{F}_k is the sum of the minimal section class and k times the fibre class. By joining x with the maximal fixed component F_{\max} by a chain of \mathbb{C}^\times -orbits, we obtain a relation $[\sigma_x] = [\sigma_0] + d$ with d a sum of \mathbb{C}^\times -invariant curves in X . \square

Let $H_2^{\text{sec}}(E, \mathbb{Z})$ denote the affine subspace of $H_2(E, \mathbb{Z})/\text{tors}$ which consists of section classes, that is the classes that project to the positive generator of $H_2(\mathbb{P}^1, \mathbb{Z})$. We set $\text{NE}(E)_{\mathbb{Z}}^{\text{sec}} := \text{NE}(E)_{\mathbb{Z}} \cap H_2^{\text{sec}}(E, \mathbb{Z})$. The above lemma shows that

$$(6) \quad \text{NE}(E)_{\mathbb{Z}}^{\text{sec}} = [\sigma_0] + \text{NE}(X)_{\mathbb{Z}}.$$

Let r be the rank of $H^2(X)$. We choose an integral basis $\{p_1, \dots, p_r\}$ of $H^2(X)$ which pairs non-negatively with $\text{NE}(X)_{\mathbb{Z}}$ (i.e. p_i is nef). There are unique lifts of p_1, \dots, p_r in $H^2(E)$ which vanish on $[\sigma_0]$. We denote these lifts by the same symbols p_1, \dots, p_r . By the above lemma, these lifts are also nef. Let $p_0 \in H^2(E)$ denote the pull-back of the positive generator of $H^2(\mathbb{P}^1, \mathbb{Z})$. Then $\{p_0, p_1, \dots, p_r\}$ forms an integral basis of $H^2(E)$. Let q_1, \dots, q_r denote the Novikov variables of X dual to the basis p_1, \dots, p_r . Similarly let q_0, q_1, \dots, q_r denote the Novikov variables of E dual to p_0, p_1, \dots, p_r . We take

$$\Lambda_X := \mathbb{Q}[[q_1, \dots, q_r]], \quad \Lambda_E := \mathbb{Q}[[q_0, q_1, \dots, q_r]]$$

to be the Novikov rings of X and E respectively. We write

$$q^d := q_1^{\langle p_1, d \rangle} \dots q_r^{\langle p_r, d \rangle} \in \Lambda_X \quad \text{for } d \in \text{NE}(X)_{\mathbb{Z}}$$

$$q^\beta := q_0^{\langle p_0, \beta \rangle} q_1^{\langle p_1, \beta \rangle} \dots q_r^{\langle p_r, \beta \rangle} \in \Lambda_E \quad \text{for } \beta \in \text{NE}(E)_{\mathbb{Z}}.$$

The small quantum cohomology ring

$$QH(X) = (H(X) \otimes_{\mathbb{Q}} \Lambda_X, \bullet)$$

is defined over the Novikov ring Λ_X . Let $\langle \dots \rangle_{g,k,d}^X$ (resp. $\langle \dots \rangle_{g,k,d}^E$) denote the genus g , degree d Gromov-Witten invariant of X (resp. E) with k insertions. We refer the reader to [4] and references therein for the definition of algebraic Gromov-Witten invariants. Since the proof of Givental's mirror theorem [8, 9] is based on algebraic geometry, we will work with algebraic Gromov-Witten invariants.

Definition 2.3. The **Seidel element** of X is the class

$$(7) \quad S := \sum_{\alpha} \sum_{\beta \in \text{NE}(E)_{\mathbb{Z}}^{\text{sec}}} \langle \iota_* \phi_{\alpha} \rangle_{0,1,\beta}^E \phi^{\alpha} q^{\beta}$$

in $QH(X) \otimes_{\Lambda_X} \Lambda_E$. Here $\iota: X \rightarrow E$ denotes the inclusion of a fibre. By Equation (6), the Seidel element can be factorized as $S = q_0 \tilde{S}$ with $\tilde{S} \in QH(X)$.

In general, one can define the Seidel element $S(\lambda)$ for a loop λ in the group $\text{Ham}(X)$ of Hamiltonian diffeomorphisms and one gets a representation of $\pi_1(\text{Ham}(X))$ on $QH(X)$ via the quantum multiplication by $S(\lambda)$. In our simple situation, this fact can be stated as follows. Suppose we have two commuting \mathbb{C}^\times -actions λ_1, λ_2 . Let $\lambda_3 = \lambda_1 * \lambda_2$ be the composite \mathbb{C}^\times -action. Let $E_i, S_i, i = 1, 2, 3$ be the X -bundle and the Seidel element associated to λ_i . The two commuting \mathbb{C}^\times -actions define the

associated X -bundle \widehat{E} over $\mathbb{P}^1 \times \mathbb{P}^1$ such that the restriction to $\mathbb{P}^1 \times \{z\}$ (resp. $\{z\} \times \mathbb{P}^1$) is isomorphic to the bundle E_1 (resp. E_2). Then E_3 can be obtained as the restriction of \widehat{E} to the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. From this geometry we have a natural map²

$$H_2^{\text{sec}}(E_1, \mathbb{Z}) \times H_2^{\text{sec}}(E_2, \mathbb{Z}) \rightarrow H_2^{\text{sec}}(E_3, \mathbb{Z}).$$

Under the multiplication of Novikov variables induced by this map, we have

$$S_3 = S_1 \bullet S_2.$$

By considering the inverse \mathbb{C}^\times -action, we find that the Seidel element is invertible in $QH(X)$ since the trivial \mathbb{C}^\times -action gives rise to the trivial Seidel element $q_0 1$.

2.2. J-functions.

Definition 2.4 ([8, 9]). The (small) **J-function** of E is the cohomology valued function

$$J_E(q, z) = e^{\sum_{i=0}^r p_i \log q_i / z} \left(1 + \sum_{\alpha=1}^M \sum_{\beta \in \text{NE}(E)_z \setminus \{0\}} \left\langle \frac{\hat{\phi}_\alpha}{z(z-\psi)} \right\rangle_{0,1,\beta}^E \hat{\phi}^\alpha q^\beta \right),$$

where z is a formal variable, and ψ is the first Chern class of the universal cotangent line bundle $\mathcal{L} \rightarrow \overline{\mathcal{M}}_{0,1}(E, d)$ at the marked point. The fraction $\hat{\phi}_\alpha / (z(z-\psi))$ in the correlator should be expanded in the series $\sum_{n \geq 0} z^{-2-n} \hat{\phi}_\alpha \psi^n$. The J-function $J_X(q, z)$ of X is defined similarly (see Equation (1)).

In order to see the relation with the Seidel elements, we expand J_E in terms of powers of z as follows.

$$J_E(q, z) = e^{\sum_{i=0}^r p_i \log q_i / z} \left(1 + z^{-2} \sum_{n=0}^{\infty} F_n(q_1, \dots, q_r) q_0^n + O(z^{-3}) \right),$$

where the functions $F_n(q_1, \dots, q_r)$ are power series with values in $H^*(E)$.

Proposition 2.5. *The Seidel element of the action is given by $S = \iota^*(F_1(q_1, \dots, q_r) q_0)$.*

Proof. From Definition 2.4 we find

$$F_n(q_1, \dots, q_r) = \sum_{\alpha=1}^M \sum_{d \in \text{NE}(X)_z} \left\langle \hat{\phi}_\alpha \right\rangle_{0,1,d+n\sigma_0}^E q^d \hat{\phi}^\alpha.$$

Using the duality identity

$$\sum_{\alpha=1}^M \hat{\phi}_\alpha \otimes \iota^* \hat{\phi}^\alpha = \sum_{\alpha=1}^N \iota_* \phi_\alpha \otimes \phi^\alpha,$$

we get

$$(8) \quad \iota^* F_1(q_1, \dots, q_r) = \sum_{\alpha=1}^N \sum_{d \in \text{NE}(X)_z} \langle i_* \phi_\alpha \rangle_{0,1,d+\sigma_0}^E q^d \phi^\alpha.$$

The conclusion follows from Equations (7) and (8). \square

²In symplectic topology, E_3 is isomorphic to the fibre sum $E_1 \# E_2$.

3. SEIDEL ELEMENTS FOR TORIC MANIFOLDS

3.1. Notation. We now fix some notation on toric geometry for this paper. For more details see [1, 4, 5]. For this paper a **toric manifold** X is a projective smooth toric variety, as constructed from the following data.

- (i) An integral lattice $M \cong \mathbb{Z}^n$ and its dual $N = \text{Hom}(M, \mathbb{Z})$. We denote by $\langle \cdot, \cdot \rangle$ the natural pairing between N and M .
- (ii) A fan Σ in $N_{\mathbb{R}} := N \otimes \mathbb{R}$ consisting of a collection of strongly convex rational polyhedral cones $\sigma \subset N_{\mathbb{R}}$, which is closed under intersections and taking faces.

We shall assume that the fan Σ is complete and regular. Let $\Sigma(1)$ denote the set of 1-cones (rays) in Σ , and we let b_i denote the set of integral primitive generators of the 1-cones. The group N is the lattice of the torus $N \otimes \mathbb{C}^\times$ and thus M is the lattice of characters in $N \otimes \mathbb{C}^\times$. The *fan sequence* of X is the exact sequence

$$(9) \quad 0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \longrightarrow N \longrightarrow 0,$$

where the second map takes the canonical basis to the primitive generators b_1, \dots, b_m and \mathbb{L} is defined to be the kernel of the second map. This in turn defines a torus $\mathbb{T} = \mathbb{L} \otimes \mathbb{C}^\times$, with character and weight lattices $\mathbb{L}, \mathbb{L}^\vee$ respectively and a sequence

$$0 \longrightarrow \mathbb{T} \longrightarrow (\mathbb{C}^\times)^m \longrightarrow N \otimes \mathbb{C}^\times \longrightarrow 0.$$

The dual of the sequence (9) is the *divisor sequence*

$$(10) \quad 0 \longrightarrow M \longrightarrow (\mathbb{Z}^m)^\vee \longrightarrow \mathbb{L}^\vee \longrightarrow 0.$$

The first arrow takes $v \in M$ into the tuple $(\langle v, b_i \rangle)_{i=1}^m$. The images of the canonical basis under the second map will be denoted by $D_i, i = 1, \dots, m$.

The weights D_i give an homomorphism $\mathbb{T} \rightarrow (\mathbb{C}^\times)^m$, and we let the torus \mathbb{T} act on \mathbb{C}^m via this homomorphism. The combinatorics of the fan defines a stability condition of this action as follows. Let $Z(\Sigma)$ denote the union

$$(11) \quad Z(\Sigma) := \bigcup_{I \in \mathcal{A}} \mathbb{C}^I, \quad \mathbb{C}^I = \{(z_1, \dots, z_m) : z_i = 0 \text{ for } i \notin I\}.$$

where \mathcal{A} is the collection of anti-cones, that is the subsets of indices that do not yield a cone in the fan

$$\mathcal{A} := \left\{ I : \sum_{i \in I} \mathbb{R}_{\geq 0} b_i \notin \Sigma \right\}.$$

The toric variety X is defined as the quotient

$$X := [\mathcal{U}/\mathbb{T}]; \quad \mathcal{U} := \mathbb{C}^m \setminus Z(\Sigma).$$

Each character $\xi : \mathbb{T} \rightarrow \mathbb{C}^\times$ defines a line bundle

$$L_\xi := \mathbb{C} \times_{\xi, \mathbb{T}} \mathcal{U} \rightarrow X.$$

The correspondence $\xi \mapsto L_\xi$ yields an identification of the Picard group with the character group of \mathbb{T} . Thus, we have

$$\mathbb{L}^\vee = \text{Hom}(\mathbb{T}, \mathbb{C}^\times) \cong \text{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}).$$

The Poincaré dual of the prime toric divisor $\{z_i = 0\} \subset X$ is the image of D_i in $H^2(X, \mathbb{Z})$. By abuse of notation, D_i denotes both the divisor $\{z_i = 0\}$ itself and its class in $H^2(X, \mathbb{Z}) \cong \mathbb{L}^\vee$. We note that $\mathbb{L} = H_2(X, \mathbb{Z})$.

The Kähler cone C_X of X , the cone consisting of Kähler classes, is given by

$$C_X := \bigcap_{I \in \mathcal{A}} \sum_{i \in I} \mathbb{R}_{>0} D_i \subset \mathbb{L}^\vee \otimes \mathbb{R} = H^2(X; \mathbb{R}).$$

We assume that C_X is nonempty so that X is projective. We will need later the following notation. As before $p_1, \dots, p_r \in H^2(X, \mathbb{Z})$ denote a nef integral basis, that is an integral basis such that $p_a \in \overline{C}_X$. Then we write the toric divisors as

$$(12) \quad D_j = \sum_{i=1}^r m_{ij} p_i,$$

for some m_{ij} . The Mori cone $\text{NE}(X) \subset H_2(X, \mathbb{R})$ is the dual of the cone \overline{C}_X . As before $\text{NE}(X)_{\mathbb{Z}}$ denotes the semi-group $\text{NE}(X) \cap H_2(X, \mathbb{Z})$.

We shall now explain the symplectic structure of X . Take $\mathbb{T}_{\mathbb{R}}$ to be maximal compact in \mathbb{T} . The $\mathbb{T}_{\mathbb{R}}$ -action on \mathbb{C}^m is generated by the Hamiltonian

$$h: \mathbb{C}^m \rightarrow \mathfrak{t}_{\mathbb{R}}^\vee, \quad h(z_1, \dots, z_m) = \sum_{i=1}^m |z_i|^2 D_i.$$

Taking a Kähler class $\eta \in C_X$, we have an homeomorphism (cf. [1, 11])

$$h^{-1}(\eta)/T_{\mathbb{R}} \cong X,$$

which induces a symplectic structure (still denoted by) η on X . This fact and the equivalence of the algebraic and symplectic Gromov-Witten invariants [13] yield the following expected result.

Lemma 3.1. *The Seidel element of the symplectic toric manifold (X, η) as defined in [19, 17] coincides with the one in Equation (7).*

3.2. The \mathbb{C}^\times -action fixing a toric divisor. For each divisor D_j we take a \mathbb{C}^\times -action on X rotating around D_j and describe the geometry of its associated bundle.

Consider the action of \mathbb{C}^\times on \mathbb{C}^m given by

$$(z_1, \dots, z_m) \mapsto (z_1, \dots, t^{-1} z_j, \dots, z_m), \quad t \in \mathbb{C}^\times$$

and the induced action on $X = (\mathbb{C}^m \setminus Z(\Sigma))/\mathbb{T}$. The toric divisor $D_j = \{z_j = 0\}$ is the maximal fixed component of this action. We extend this to the diagonal \mathbb{C}^\times -action on $(\mathbb{C}^m \setminus Z(\Sigma)) \times (\mathbb{C}^2 \setminus \{0\})$ by

$$(z_1, \dots, z_m, u, v) \mapsto (z_1, \dots, t^{-1} z_j, \dots, z_m, tu, tv), \quad t \in \mathbb{C}^\times.$$

The associated bundle E_j of the \mathbb{C}^\times -action on X is given by

$$E_j = (\mathbb{C}^m \setminus Z(\Sigma)) \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{T} \times \mathbb{C}^\times.$$

Therefore E_j is also a toric variety. We can identify $H^2(E_j, \mathbb{Z})$ with the lattice of characters of $\mathbb{T} \times \mathbb{C}^\times$:

$$(13) \quad H^2(E_j, \mathbb{Z}) \cong \mathbb{L}^\vee \oplus \mathbb{Z} \cong H^2(X, \mathbb{Z}) \oplus \mathbb{Z}.$$

This is dual to the splitting in Equation (5). In light of this splitting, the $m+2$ weights of $\mathbb{T} \times \mathbb{C}^\times$ defining E_j are just given by

$$(14) \quad \widehat{D}_i = (D_i, 0) \text{ for } i \neq j; \quad \widehat{D}_j = (D_j, -1); \quad \widehat{D}_{m+1} = \widehat{D}_{m+2} = (\vec{0}, 1).$$

This in turn yields the divisor sequence

$$0 \longrightarrow M \oplus \mathbb{Z} \longrightarrow (\mathbb{Z}^{m+2})^\vee \xrightarrow{\widehat{D}} \mathbb{L}^\vee \oplus \mathbb{Z} \longrightarrow 0.$$

The fan of E_j is contained in $N_{\mathbb{R}} \oplus \mathbb{R}$. The following generators of the 1-cones

$$\hat{b}_i = (b_i, 0) \text{ for } 1 \leq i \leq m; \quad \hat{b}_{m+1} = (\vec{0}, 1); \quad \hat{b}_{m+2} = (b_j, -1),$$

yield the fan sequence for E_j

$$0 \longrightarrow \mathbb{L} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}^{m+2} \xrightarrow{\hat{b}} N \oplus \mathbb{Z} \longrightarrow 0$$

which is dual to the divisor sequence above. We set

$$p_0 := (\vec{0}, 1) = \hat{D}_{m+1} = \hat{D}_{m+2} \in H^2(E_j).$$

Under the splitting (13), a nef integral basis $\{p_1, \dots, p_r\}$ of $H^2(X)$ can be lifted to a nef integral basis $\{p_0, p_1, \dots, p_r\}$ of $H^2(E_j)$. We have

$$C_{E_j} = C_X + \mathbb{R}_{>0} p_0, \quad c_1(E_j) = c_1(X) + p_0.$$

The following result is immediate.

Lemma 3.2. *If $-K_X$ is nef then for all j , $-K_{E_j}$ is nef.*

3.3. I functions and the mirror maps. We now recall Givental's mirror Theorem [9, Theorem 0.1]. Let $\{p_0, p_1, \dots, p_r\}$ be a nef basis of $H^2(E_j)$ as above. The I-function of X is the $H^*(X)$ -valued function:

$$I_X(y, z) = e^{\sum_{i=1}^r p_i \log y_i / z} \sum_{d \in \text{NE}(X)_{\mathbb{Z}}} \prod_{i=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_i + kz)}{\prod_{k=-\infty}^{\langle D_i, d \rangle} (D_i + kz)} \right) y^d.$$

Note that all but finitely many factors in the infinite products cancel. Here $y^d = y_1^{\langle p_1, d \rangle} \dots y_r^{\langle p_r, d \rangle}$. Similarly the I-function of E_j is the $H^*(E_j)$ -valued function

$$(15) \quad I_{E_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \sum_{\beta \in \text{NE}(E_j)_{\mathbb{Z}}} \prod_{i=1}^{m+2} \left(\frac{\prod_{k=-\infty}^0 (\hat{D}_i + kz)}{\prod_{k=-\infty}^{\langle \hat{D}_i, \beta \rangle} (\hat{D}_i + kz)} \right) y^{\beta},$$

where $y^{\beta} = y_0^{\langle p_0, \beta \rangle} y_1^{\langle p_1, \beta \rangle} \dots y_r^{\langle p_r, \beta \rangle}$.

Theorem 3.3 (Givental [9]). *Let X be a toric manifold with $-K_X$ nef. Then we have*

$$I_X(y, z) = J_X(q, z)$$

under an invertible change of variables of the form

$$(16) \quad \log q_i = \log y_i + g_i(y_1, \dots, y_r), \quad i = 1, \dots, r$$

where $g_i(y)$ is a power series in y_1, \dots, y_r which is homogeneous of degree zero with respect to the degree $\deg y^d = 2 \langle c_1(X), d \rangle$ and $g_i(0) = 0$.

Definition 3.4. The coordinates y_1, \dots, y_r are called the **mirror coordinates** of X . They are asymptotically the same as q_1, \dots, q_r as $y \rightarrow 0$, in the sense that $q_i = y_i + \text{higher order terms}$.

Because $c_1(E_j)$ is nef, we can apply this mirror theorem to E_j . Hence we have

$$I_{E_j}(y, z) = J_{E_j}(q, z)$$

under a change of variables

$$(17) \quad \log q_i = \log y_i + g_i^{(j)}(y_0, y_1, \dots, y_r), \quad i = 0, \dots, r.$$

Lemma 3.5. *The function $g_i^{(j)}$ does not depend on y_0 . Moreover we have*

$$g_i^{(j)}(y_0, y_1, \dots, y_r) = g_i(y_1, \dots, y_r), \quad i = 1, \dots, r.$$

This means that the mirror maps for X and E_j coincide for q_1, \dots, q_r .

Proof. The functions $g_i^{(j)}$ appear as the coefficients of z^{-1} in the expansion of I_{E_j} (see [9, p.145]):

$$(18) \quad I_{E_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \left(1 + z^{-1} \sum_{i=0}^r g_i^{(j)}(y) p_i + O(z^{-2}) \right).$$

The functions g_i are determined by I_X similarly. Using $\iota^* \widehat{D}_j = D_j$, we can see that

$$(19) \quad \iota^* I_{E_j} \Big|_{y_0=0} = I_X.$$

Note that the restriction to $y_0 = 0$ in the left-hand side is well defined because $\iota^* p_0 = 0$. These facts imply that

$$g_i^{(j)}(0, y_1, \dots, y_r) = g_i(y_1, \dots, y_r), \quad i = 1, \dots, r.$$

On the other hand, $\deg y_0 = 2 \langle c_1(E_j), \sigma_0 \rangle = 2$ and all other monomials appearing in $g_i^{(j)}$ have nonnegative degree (since $c_1(E_j)$ is nef). Thus the homogeneous series $g_i^{(j)}$ of degree zero does not depend on y_0 . \square

3.3.1. Batyrev relations and elements. Let y_1, \dots, y_r be the mirror coordinates of a toric manifold X with $-K_X$ nef. Set $\mathbb{Q}[y^\pm] = \mathbb{Q}[y_1^\pm, \dots, y_r^\pm]$. Batyrev's quantum ring is a $\mathbb{Q}[y^\pm]$ -algebra generated by the variables w_1, \dots, w_m corresponding to the toric divisors D_1, \dots, D_m subject to the following two types of relations:

$$(20) \quad \begin{aligned} & \text{(multiplicative):} \quad \prod_{j: \langle D_j, d \rangle > 0} w_j^{\langle D_j, d \rangle} = y^d \prod_{j: \langle D_j, d \rangle < 0} w_j^{-\langle D_j, d \rangle}, \quad d \in H_2(X, \mathbb{Z}); \\ & \text{(linear):} \quad \sum_{j=1}^m c_j w_j = 0 \quad \text{whenever} \quad \sum_{j=1}^m c_j D_j = 0, \quad c_j \in \mathbb{Q}. \end{aligned}$$

where $y^d = \prod_{i=1}^r y_i^{\langle p_i, d \rangle}$. We refer to these relations as **Batyrev relations**. By the divisor sequence (10), the linear relations can be written in the form:

$$(21) \quad \sum_{j=1}^m \langle v, b_j \rangle w_j = 0, \quad v \in M.$$

Remark 3.6. Since X is compact, there exist positive integers c_1, \dots, c_m such that $c_1 b_1 + \dots + c_m b_m = 0$. Then by the fan sequence (9) we have $d \in H_2(X, \mathbb{Z})$ such that $\langle D_i, d \rangle = c_i > 0$. This gives a relation $\prod_{i=1}^m w_i^{c_i} = y^d$. Therefore the variables w_i are invertible in the Batyrev ring.

Definition 3.7. Let X be a toric manifold with $-K_X$ nef. We can regard $p_i \in H^2(X)$ as corresponding to the logarithmic vector field $q_i(\partial/\partial q_i)$. We introduce an element $\tilde{p}_i \in H^2(X) \otimes \mathbb{Q}[[y_1, \dots, y_r]]$ which corresponds to $y_i(\partial/\partial y_i)$ as

$$\tilde{p}_i = \sum_{k=1}^r \frac{\partial \log q_k}{\partial \log y_i} p_k = p_i + O(y).$$

Recall from Equation (12) that $D_j = \sum_{i=1}^r m_{ij} p_i$. We define the **Batyrev element** associated to D_j as

$$\tilde{D}_j = \sum_{i=1}^r m_{ij} \tilde{p}_i = D_j + O(y).$$

Proposition 3.8. *The Batyrev elements $\tilde{D}_1, \dots, \tilde{D}_m$ satisfy both the multiplicative and linear Batyrev relations (20) for $w_j = \tilde{D}_j$.*

Proof. We use the fact [8, 9, 4] that if the J-function satisfies the differential equation $R(q_i, zq_i(\partial/\partial q_i), z)J_X(q, z) = 0$, then we have a relation $R(q_i, p_i \bullet, 0)1 = 0$ in the small quantum cohomology ring. We can easily show that the I-function of X satisfy the differential equation $R_d I_X(y, z) = 0$ for $d \in H_2(X, \mathbb{Z})$ where

$$R_d = \prod_{j: \langle D_j, d \rangle > 0} \prod_{k=0}^{\langle D_j, d \rangle - 1} (\mathcal{D}_j - kz) - y^d \prod_{j: \langle D_j, d \rangle < 0} \prod_{k=0}^{-\langle D_j, d \rangle - 1} (\mathcal{D}_j - kz)$$

and $\mathcal{D}_j = z \sum_{i=1}^r m_{ij} y_i (\partial/\partial y_i)$. From the mirror theorem, we know that the J-function satisfies the corresponding differential equation under the mirror change of coordinates. The multiplicative Batyrev relations follow from this and the fact above. It is obvious that \tilde{D}_j 's satisfy the linear relations. \square

Remark 3.9. Seidel elements satisfy the multiplicative relations (2) in the q -coordinates, as proved in [17, Proposition 5.2], but do not necessarily satisfy the linear relations.

3.4. Seidel elements in terms of mirror maps. In this paragraph we will assume that $-K_X$ is nef. Since $-K_{E_j}$ is also nef, we can expand the I-function of E_j in z^{-1} as follows (cf. (18)).

$$(22) \quad I_{E_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \left(1 + z^{-1} \sum_{i=0}^r g_i^{(j)}(y) p_i + z^{-2} \sum_{n=0}^2 G_n^{(j)}(y) y_0^n + O(z^{-3}) \right)$$

where $g_i^{(j)}(y)$ is the mirror map of E_j in Equation (17). The coefficients $G_0^{(j)}, G_1^{(j)}, G_2^{(j)}$ are power series in y_1, \dots, y_r taking values in $H^4(E_j), H^2(E_j), H^0(E_j)$ respectively. Under the coordinate change $\log q_i = \log y_i + g_i^{(j)}(y)$, we can rewrite $I_{E_j}(y, z)$ as

$$e^{\sum_{i=0}^r p_i \log q_i / z} \left(1 + z^{-2} \left(G_0^{(j)} - \frac{1}{2} \left(\sum_{i=0}^r g_i^{(j)} p_i \right)^2 + G_1^{(j)} y_0 + G_2^{(j)} y_0^2 \right) + O(z^{-3}) \right).$$

Lemma 3.10. *The Seidel element S_j associated to the toric divisor D_j is given by*

$$S_j(q_0, \dots, q_r) = \iota^*(G_1^{(j)}(y_1, \dots, y_r) y_0),$$

under the mirror transformation (17) for E_j .

Proof. By Proposition 2.5, the Seidel element \tilde{S}_j is the coefficient of q_0/z^2 in $\exp(-\sum_{i=0}^r p_i \log q_i / z) J_{E_j}(q, z)$. The result follows from $J_{E_j}(q, z) = I_{E_j}(y, z)$. \square

We now digress to reinterpret $G_1^{(j)}$. A straightforward computation shows the following lemma.

Lemma 3.11. *The I -function of E_j satisfies the differential equation*

$$z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{E_j} = \left(\sum_{i=1}^r m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{E_j}.$$

where (m_{ij}) is the matrix appearing in Equation (12).

Using the expansion for I_{E_j} in Equation (22) we obtain

$$\begin{aligned} z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{E_j} &= \\ &= \frac{\partial}{\partial y_0} \left(e^{\sum_{i=0}^r p_i \log y_i / z} \left(p_0 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)} p_i p_0 + \sum_{n=0}^2 G_n^{(j)} n y_0^n \right) + O(z^{-2}) \right) \right) \\ (23) \quad &= e^{\sum_{i=0}^r p_i \log y_i / z} \left(z^{-1} \sum_{n=1}^2 G_n^{(j)} n^2 y_0^{n-1} + O(z^{-2}) \right) \end{aligned}$$

where we used $p_0^2 = 0$. On the other hand

$$\begin{aligned} \iota^* \left(\sum_{i=1}^r m_{ij} y_i \frac{\partial}{\partial y_i} - y_0 \frac{\partial}{\partial y_0} \right) I_{E_j} &= \left(\sum_{i=1}^r m_{ij} y_i \frac{\partial}{\partial y_i} - y_0 \frac{\partial}{\partial y_0} \right) \iota^* I_{E_j} \\ &= \left(\sum_{i=1}^r m_{ij} y_i \frac{\partial}{\partial y_i} \right) (I_X + O(y_0)) \quad \text{by Equation (19)} \\ &= \left(\sum_{i=1}^r m_{ij} y_i \frac{\partial}{\partial y_i} \right) (J_X + O(q_0)) \quad \text{by mirror Theorem 3.3} \\ &= \left(\sum_{i=1}^r \sum_{k=1}^r m_{ij} \frac{\partial \log q_k}{\partial \log y_i} \frac{\partial}{\partial \log q_k} \right) e^{\sum_{i=1}^r p_i \log q_i / z} (1 + O(z^{-2}) + O(q_0)) \\ (24) \quad &= e^{\sum_{i=1}^r p_i \log q_i / z} (z^{-1} \tilde{D}_j + O(z^{-2}) + O(q_0)) \end{aligned}$$

where \tilde{D}_j is the Batyrev element of D_j . Here y_0, \dots, y_r and q_0, \dots, q_r are related by the mirror map (17) for E_j , but we know by Lemma 3.5 that the mirror maps of X and E_j coincide for q_1, \dots, q_r . Comparing (24) and (23) we get the following result.

Lemma 3.12. $\tilde{D}_j(y) = \iota^* G_1^{(j)}(y_1, \dots, y_r)$.

Noting $S_j = q_0 \tilde{S}_j$ and the effect of the mirror map $\log q_0 = \log y_0 + g_0^{(j)}(y)$, we obtain the desired expression of Seidel elements from Lemma 3.12 and Lemma 3.10.

Theorem 3.13. *The Seidel element \tilde{S}_j and Batyrev element \tilde{D}_j are related by*

$$\tilde{S}_j(q_1, \dots, q_r) = \exp \left(-g_0^{(j)}(y_1, \dots, y_r) \right) \tilde{D}_j(y_1, \dots, y_r)$$

under the mirror transformation (16) of X .

Remark 3.14. The theorem above and Remark 3.6 show the invertibility of the Seidel elements, obtaining the result of Seidel in our particular toric setting.

We shall calculate $g_0^{(j)}$ and \tilde{D}_j as explicit hypergeometric series in the mirror coordinates y . It is not hard to see the following lemma.

Lemma 3.15. *About the product factors appearing in the I-function I_{E_j} (15), we have*

$$\prod_{i=1}^{m+2} \frac{\prod_{k=-\infty}^0 (\widehat{D}_i + kz)}{\prod_{k=-\infty}^{\langle \widehat{D}_i, \beta \rangle} (\widehat{D}_i + kz)} = C_\beta z^{-\sum_{i=1}^{m+2} \langle \widehat{D}_i, \beta \rangle - \#\{i : \langle \widehat{D}_i, \beta \rangle < 0\}} \prod_{i : \langle \widehat{D}_i, \beta \rangle < 0} \widehat{D}_i + h.o.t.$$

where *h.o.t.* means higher order terms in z^{-1} and

$$(25) \quad C_\beta = \prod_{i : \langle \widehat{D}_i, \beta \rangle < 0} (-1)^{-\langle \widehat{D}_i, \beta \rangle - 1} (-\langle \widehat{D}_i, \beta \rangle - 1)! \cdot \prod_{i : \langle \widehat{D}_i, \beta \rangle \geq 0} (\langle \widehat{D}_i, \beta \rangle!)^{-1}.$$

Lemma 3.16. *The coefficient $g_0^{(j)}$ is given by*

$$(26) \quad g_0^{(j)}(y_1, \dots, y_r) = \sum_{\substack{\langle c_1(X), d \rangle = 0 \\ \langle D_j, d \rangle < 0 \\ \langle D_i, d \rangle \geq 0, \forall i \neq j}} \frac{(-1)^{\langle D_j, d \rangle} (-\langle D_j, d \rangle - 1)!}{\prod_{i \neq j} \langle D_i, d \rangle!} y^d.$$

Proof. We want to investigate the coefficient of z^{-1} in the power series expansion (15) of I_{E_j} . By Lemma 3.15, the summand indexed by $\beta \in \text{NE}(E_j)_\mathbb{Z}$ contributes to the coefficient of z^{-1} if $\sum_{i=1}^{m+2} \langle \widehat{D}_i, \beta \rangle + \#\{i : \langle \widehat{D}_i, \beta \rangle < 0\} \leq 1$. Since $-K_{E_j} = \sum_{i=1}^{m+2} \widehat{D}_i$ is nef (Lemma 3.2), this happens only in the following three cases:

- $\sum_{i=1}^{m+2} \langle \widehat{D}_i, \beta \rangle = 0$ and $\#\{i : \langle \widehat{D}_i, \beta \rangle < 0\} = 0$;
- $\sum_{i=1}^{m+2} \langle \widehat{D}_i, \beta \rangle = 1$ and $\#\{i : \langle \widehat{D}_i, \beta \rangle < 0\} = 0$;
- $\sum_{i=1}^{m+2} \langle \widehat{D}_i, \beta \rangle = 0$ and $\#\{i : \langle \widehat{D}_i, \beta \rangle < 0\} = 1$.

In the first case, we have $\langle \widehat{D}_i, \beta \rangle = 0$ for all i and so $\beta = 0$. This contributes nothing to the coefficient of z^{-1} . The second case does not happen because in this case β has to satisfy $\langle \widehat{D}_i, \beta \rangle = 0$ except for one i , and this implies $\beta = 0$. In the third case, β has to be a fibre class from $\text{NE}(X)_\mathbb{Z}$ (i.e. $\langle p_0, \beta \rangle = 0$) because $\sum_{i=1}^{m+2} \widehat{D}_i = -K_X + p_0$ and $-K_X, p_0$ are nef. Therefore the coefficient of z^{-1} in I_{E_j} is the sum of

$$(27) \quad C_d \prod_{i : \langle \widehat{D}_i, d \rangle < 0} \widehat{D}_i, \quad \text{where } C_d \text{ is the constant in (25)}$$

over all the fibre classes $d \in \text{NE}(X)_\mathbb{Z}$ such that $\sum_{i=1}^{m+2} \langle \widehat{D}_i, d \rangle = \sum_{i=1}^m \langle D_i, d \rangle = 0$ and $\langle \widehat{D}_i, d \rangle = \langle D_i, d \rangle < 0$ for exactly one i from $\{1, \dots, m\}$. (Note that $\langle D_{m+1}, d \rangle = \langle D_{m+2}, d \rangle = 0$.)

Now $g_0^{(j)}$ is the coefficient corresponding to p_0 . Among the divisors $\widehat{D}_1, \dots, \widehat{D}_m$, $\widehat{D}_j = (D_j, -1) = D_j - p_0$ is the only one which contains p_0 . Therefore the terms of the form (27) which contribute to $g_0^{(j)}$ are those with $d \in \text{NE}(X)_\mathbb{Z}$ for which $\langle D_j, d \rangle < 0$, $\langle D_i, d \rangle \geq 0$ for $i \neq j$ and $\sum_{i=1}^m \langle D_i, d \rangle = 0$. \square

Lemma 3.17. *The Batyrev element \widetilde{D}_j is given by*

$$\widetilde{D}_j = D_j - \sum_{i=1}^m D_i \sum_{\substack{\langle c_1(X), d \rangle = 0 \\ \langle D_i, d \rangle < 0 \\ \langle D_k, d \rangle \geq 0, \forall k \neq i}} (-1)^{\langle D_i, d \rangle} \frac{(-\langle D_i, d \rangle - 1)!}{\prod_{k \neq i, j} \langle D_k, d \rangle!} y^d.$$

Proof. By the same calculation leading to Equation (24), we find

$$\begin{aligned} \left(\sum_{i=1}^r m_{ij} y_i \frac{\partial}{\partial y_i} \right) I_X(y, z) &= e^{\sum_{i=1}^r p_i \log q_i / z} \left(z^{-1} \tilde{D}_j + O(z^{-2}) \right) \\ &= e^{\sum_{i=1}^r p_i \log y_i / z} \left(z^{-1} \tilde{D}_j + O(z^{-2}) \right). \end{aligned}$$

The conclusion follows from a calculation similar to the previous lemma. \square

3.5. Example. Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ be the second Hirzebruch surface. Let p_1, p_2 be the nef basis of $H^2(X)$ that are Poincaré dual to the fibre and the infinity section. The divisor matrix is

$$(m_{ij}) = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

That is

$$D_1 = p_2, \quad D_2 = p_2 - 2p_1, \quad D_3 = D_4 = p_1.$$

The q coordinates are

$$q_1 = e^{\text{PD}(D_2)}, \quad q_2 = e^{\text{PD}(D_3)}.$$

The mirror transformation of X is well known (cf. [9, p.146]); it is given by

$$\begin{aligned} y_1 &= \frac{q_1}{(1 + q_1)^2}, \\ y_2 &= q_2(1 + q_1). \end{aligned}$$

The classes \tilde{p}_i , $i = 1, 2$ corresponding to the vector fields $y_i(\partial/\partial y_i)$ are

$$\begin{aligned} \tilde{p}_1 &= \frac{1 + q_1}{1 - q_1} p_1 - \frac{q_1}{1 - q_1} p_2, \\ \tilde{p}_2 &= p_2. \end{aligned}$$

The Batyrev elements are

$$\tilde{D}_1 = \tilde{p}_2, \quad \tilde{D}_2 = \tilde{p}_2 - 2\tilde{p}_1, \quad \tilde{D}_3 = \tilde{D}_4 = \tilde{p}_1.$$

Hence

$$\begin{aligned} \tilde{D}_1 &= D_1, \\ \tilde{D}_2 &= \frac{1 + q_1}{1 - q_1} D_2, \\ \tilde{D}_3 &= D_3 - \frac{q_1}{1 - q_1} D_2, \\ \tilde{D}_4 &= D_4 - \frac{q_1}{1 - q_1} D_2. \end{aligned}$$

The correction term $g_0^{(j)}$ appears only for $j = 2$. By Theorem 3.13, the Seidel element associated to D_2 is just given by

$$\tilde{S}_2(q_1, q_2) = \exp \left(-g_0^{(2)}(y_1, y_2) \right) \tilde{D}_2(y_1, y_2).$$

where the term $g_0^{(2)}$ is the sum over all effective classes d such that $\langle c_1(X), d \rangle = 0$, $\langle D_j, d \rangle < 0$, $\langle D_i, d \rangle \geq 0$, $i \neq j$ as in Equation (26). These are just the classes

$d = d_1 \text{PD}(D_2)$, for $d_1 \geq 0$. So we have $\langle D_3, d \rangle = \langle D_4, d \rangle = d_1, \langle D_1, d \rangle = 0$ and $\langle D_2, d \rangle = -2d_1$. Thus

$$g_0(y) = \sum_{d_1=1}^{\infty} \frac{(2d_1-1)!}{(d_1!)^2} y_1^{d_1}.$$

(This has already appeared in other places e.g. [4, p.393] or [9, p.146].) Using the formulas above, it is not hard to see that

$$\exp(-g_0) = (1 + q_1)^{-1},$$

and thus $\tilde{S}_2 = (1 + q_1)^{-1} \tilde{D}_2$. All other Seidel elements \tilde{S}_i agree with the Batyrev elements \tilde{D}_i . Therefore we have,

$$\begin{aligned} \tilde{S}_1 &= D_1, \\ \tilde{S}_2 &= \left(\frac{1}{1 - q_1} \right) D_2, \\ \tilde{S}_3 &= D_3 - \left(\frac{q_1}{1 - q_1} \right) D_2, \\ \tilde{S}_4 &= D_4 - \left(\frac{q_1}{1 - q_1} \right) D_2. \end{aligned}$$

This computation agrees with the one in McDuff-Tolman [17].

It is easy to check that the Batyrev relations are compatible in the two coordinate systems

$$\begin{aligned} \tilde{D}_1^{d_2} \tilde{D}_2^{d_2-2d_1} \tilde{D}_3^{d_1} \tilde{D}_4^{d_1} &= y_1^{d_1} y_2^{d_2}, \\ \tilde{S}_1^{d_2} \tilde{S}_2^{d_2-2d_1} \tilde{S}_3^{d_1} \tilde{S}_4^{d_1} &= q_1^{d_1} q_2^{d_2}. \end{aligned}$$

4. RECONSTRUCTION OF MIRROR MAPS

4.1. When the correction term vanishes. We continue to assume that X is a toric manifold with $-K_X$ nef. Let Σ denote its fan. If the toric divisor D_j is nef, then a direct computation from Equation (26) shows that the correction coefficient $g_0^{(j)}$ is trivial. In such case the Seidel element and Batyrev element agree. However the nef condition is too restrictive. We show that $g_0^{(j)}$ vanishes if and only if the restriction of $-K_X$ to D_j is big, i.e. the image of D_j under the map $\phi_{|-mK_X|}: X \rightarrow \mathbb{P}(H^0(X, -mK_X))$ has the same dimension as D_j for sufficiently big $m > 0$.

Definition 4.1. The **fan polytope** $P \subset N \otimes \mathbb{R}$ of X is the convex hull of the integral primitive generators b_1, \dots, b_m of 1-cones of the fan Σ .

Since X is a compact toric manifold with $-K_X$ nef, we have

Lemma 4.2. *The fan polytope P of X contains the origin in its interior and every vector b_i is on the boundary of P .*

Proposition 4.3. *The following are equivalent:*

- (i) *The correction term $g_0^{(j)}$ in (26) vanishes;*
- (ii) *The primitive generator b_j is a vertex of the fan polytope P ;*
- (iii) *The anticanonical divisor $-K_X$ is big on D_j , i.e. $(-K_X)^{n-1} \cdot D_j > 0$.*

For the proof, we use the following notation and lemma.

Notation 4.4. For each cone $\sigma \in \Sigma$ we let $F(\sigma)$ denote the minimal face of the fan polytope P which contains the collection $\sigma(1)$ of primitive generators $b_i \in \sigma$. By abuse of notation we write $F(b_i)$ to denote the face $F(\mathbb{R}_{\geq 0}b_i)$.

Lemma 4.5. *Let σ be a cone in Σ . Suppose that $d \in H_2(X)$ satisfies $\langle c_1(X), d \rangle = 0$ and*

$$\langle D_i, d \rangle \geq 0 \quad \text{if } b_i \notin \sigma.$$

Then $d \in \text{NE}(X)$ and

$$\langle D_i, d \rangle = 0 \quad \text{if } b_i \notin F(\sigma).$$

Proof. Take a support function $h: N_{\mathbb{R}} \rightarrow \mathbb{R}$ such that P is contained in the half-space $\{v \in N_{\mathbb{R}} | h(v) \leq 1\}$ and such that $F(\sigma) = P \cap h^{-1}(1)$. We have the following relation

$$0 = \sum_{i=1}^m \langle D_i, d \rangle b_i.$$

By evaluating h we have

$$\begin{aligned} 0 &= \sum_{i=1}^m \langle D_i, d \rangle h(b_i) = \sum_{b_i \in F(\sigma)} \langle D_i, d \rangle + \sum_{b_i \notin F(\sigma)} \langle D_i, d \rangle h(b_i) \\ &\leq \sum_{b_i \in F(\sigma)} \langle D_i, d \rangle + \sum_{b_i \notin F(\sigma)} \langle D_i, d \rangle = 0, \end{aligned}$$

where the second inequality follows from the fact that $h(b_i) < 1$ and $\langle D_i, d \rangle \geq 0$, if $b_i \notin F(\sigma)$. Because of the zeroes in each hand-side, the inequality is an equality. Therefore $\langle D_i, d \rangle h(b_i) = \langle D_i, d \rangle$, if $b_i \notin F(\sigma)$. This in turn implies that $\langle D_i, d \rangle = 0$, if $b_i \notin F(\sigma)$. \square

Proof of Proposition 4.3. We first prove the equivalence of (ii) and (iii). By [5, Lemma 9.3.9], the bigness of $(-K_X)|_{D_j}$ translates to the fact that the face F_j^* of the dual polytope P^* has the maximal dimension $n - 1$, where

$$\begin{aligned} P^* &= \{v \in M_{\mathbb{R}} : \langle v, b_i \rangle \geq -1, \forall i\}, \\ F_j^* &= \{v \in P^* : \langle v, b_j \rangle = -1\}. \end{aligned}$$

This is equivalent to b_j being a vertex of P .

Next we prove the equivalence of (i) and (ii). Suppose b_j is not a vertex of P . Then b_j is in the relative interior of $F(b_j)$. Since $F(b_j)$ is convex, there exist b_{i_1}, \dots, b_{i_k} on $F(b_j)$ and nonnegative constants c_1, \dots, c_k such that $b_j = \sum_{s=1}^k c_s b_{i_s}$ for all s and

$$(28) \quad c_1 b_{i_1} + \dots + c_k b_{i_k} - b_j = 0,$$

$$(29) \quad c_1 + \dots + c_k - 1 = 0.$$

By the fan sequence (9), the relation (28) gives an element in $d \in \text{NE}(X)$ such that $\langle D_j, d \rangle = -1, \langle D_i, d \rangle \geq 0, i \neq j$. By Equation (29), $\langle c_1(X), d \rangle = 0$. Such d contributes to the sum in Equation (26) and $g_0^{(j)} \neq 0$.

Suppose that b_j is a vertex of P and $g_0^{(j)} \neq 0$. Then by Equation (26) there exists $d \in \text{NE}(X)$ such that $\langle D_j, d \rangle < 0, \langle D_i, d \rangle \geq 0, i \neq j$ and $\langle c_1(X), d \rangle = 0$. By Lemma 4.5, we know that $\langle D_i, d \rangle = 0$, if $b_i \notin F(b_j)$. However, b_j is a vertex of P , which means that $F(b_j)$ contains only b_j . Therefore $\langle D_i, d \rangle = 0$ for all $i \neq j$. Since

the divisors $\{D_i : i \neq j\}$ span $H^2(X, \mathbb{Q})$ it follows that $d = 0$, which contradicts $\langle D_i, d \rangle < 0$. \square

Corollary 4.6. *Let n be the dimension of X and r be the Picard number. Out of $m = n + r$ correction terms $g_0^{(j)}$, at least $n + 1$ vanish. In other words, there are at most $r - 1$ non-vanishing correction terms.*

Proof. Any convex polyhedron with non-empty interior in an n -dimensional vector space has at least $n + 1$ vertices. The result follows. \square

4.2. Reconstruction. By Proposition 3.8 the elements \tilde{D}_j satisfy both the multiplicative and the linear Batyrev relations. However Seidel elements only satisfy the multiplicative relations. The reconstruction of the mirror transformation is based on this observation. We now have all we need to establish the proof of Theorem 1.2.

Proof of Theorem 1.2. By all our statements above, Batyrev elements satisfy the conditions (i) – (iii). So we only need to show the uniqueness property. The linear relations (iii) are equivalent to the identity (cf. Equation (21))

$$\sum_{j=1}^m b_j \otimes \tilde{D}_j = 0$$

in the tensor product $N_{\mathbb{Q}} \otimes H^2(X; \Lambda_X)$. Substituting \tilde{D}_j with $H_j \tilde{S}_j$ and using the property (ii), we have

$$\sum_{b_j : \text{non-vertex}} H_j(b_j \otimes \tilde{S}_j) = - \sum_{b_j : \text{vertex}} b_j \otimes \tilde{S}_j.$$

Here we used the fact that $(-K_X)^{n-1} \cdot D_j > 0$ is equivalent to b_j being a vertex of the fan polytope (Proposition 4.3). Since $\tilde{S}_j = D_j + O(q)$, to see the uniqueness of H_j , it suffices to show that $\{b_j \otimes D_j : b_j : \text{non-vertex}\}$ is linearly independent. Suppose we have a linear relation

$$(30) \quad \sum_{b_j : \text{non-vertex}} h_j(b_j \otimes D_j) = 0.$$

If b_j is not a vertex, the face $F(b_j)$ of the fan polytope P contains b_j in its relative interior. Since $F(b_j)$ is a convex hull of vertices b_{i_1}, \dots, b_{i_k} of P , we have a relation

$$c_1 b_{i_1} + \dots + c_k b_{i_k} - b_j = 0$$

for some rational numbers c_s . This relation gives an element $d \in H_2(X)$ such that $\langle D_j, d \rangle = -1$, $\langle D_{i_s}, d \rangle = c_s$ and $\langle D_l, d \rangle = 0$ when l is none of j, i_1, \dots, i_k . Contracting Equation (30) with $\text{id} \otimes d$, we get

$$0 = -h_j b_j.$$

Thus $h_j = 0$ for all j for which b_j is not a vertex. This completes the proof. \square

4.3. Examples. We compute the Batyrev and the Seidel elements in several examples and illustrate the reconstruction of mirror maps.

4.3.1. *Hirzebruch surface \mathbb{F}_2 .* We revisit the example in Section 3.5. The fan of \mathbb{F}_2 is given by the following primitive vectors of 1-cones:

$$b_1 = (0, -1), \quad b_2 = (0, 1), \quad b_3 = (-1, 1), \quad b_4 = (1, 1).$$

Here b_1, b_3, b_4 are vertices of the fan polytope and thus $\tilde{S}_j = \tilde{D}_j$ for $j = 1, 3, 4$. The mirror coordinates y_1, y_2 can be reconstructed from these Seidel elements, via the formulas

$$\tilde{S}_1 = \sum_{i=1}^2 \frac{\partial \log q_i}{\partial \log y_2} p_i, \quad \tilde{S}_3 = \tilde{S}_4 = \sum_{i=1}^2 \frac{\partial \log q_i}{\partial \log y_1} p_i.$$

4.3.2. *Crepant resolution of $\mathbb{P}^3/\mathbb{Z}^2$.* Let $X = \mathbb{P}(\mathcal{O}(2, -2) \oplus \mathcal{O})$ be the \mathbb{P}^1 -bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Collapsing the zero and the infinity section to \mathbb{P}^1 yields $\mathbb{P}^3/\mathbb{Z}_2$, where \mathbb{Z}_2 acts on \mathbb{P}^3 by

$$[z_1 : z_2 : z_3 : z_4] \mapsto [z_1 : z_2 : -z_3 : -z_4].$$

The toric variety $\mathbb{P}^3/\mathbb{Z}_2$ has transversal A_1 singularities along two \mathbb{P}^1 's and X is its crepant resolution.

The fan of X is given by

$$\begin{aligned} b_1 &= (1, 0, -1), \quad b_2 = (-1, 0, -1), \quad b_3 = (0, 1, 1), \quad b_4 = (0, -1, 1), \\ b_5 &= (0, 0, 1), \quad b_6 = (0, 0, -1). \end{aligned}$$

The vertices of the fan polytope are b_1, b_2, b_3, b_4 . The Mori cone is spanned by the three homology classes $\gamma_1, \gamma_2, \gamma_3 \in H_2(X, \mathbb{Z})$ with the intersection matrix

$$(\langle D_i, \gamma_j \rangle)^T = \begin{pmatrix} 0 & 0 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

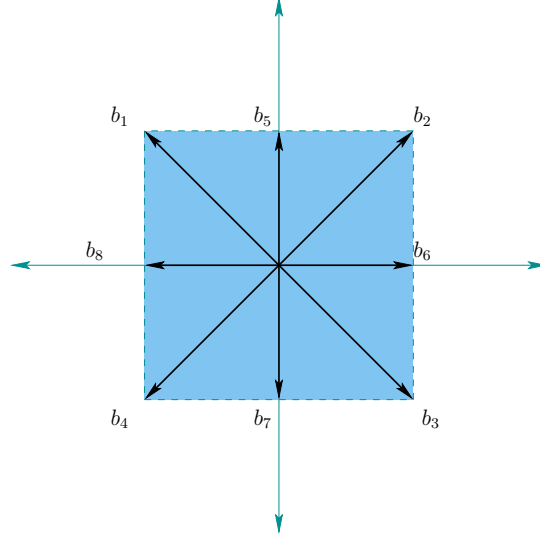
where each row vector gives a relation amongst b_1, \dots, b_6 . The classes $\gamma_1, \gamma_2, \gamma_3$ define a \mathbb{Z} -basis for $H_2(X, \mathbb{Z})$. Let $p_1, p_2, p_3 \in H^2(X, \mathbb{Z})$ denote the dual basis and let q_1, q_2, q_3 denote the corresponding Novikov variables.

Mirror coordinates.

$$\begin{aligned} y_1 &= \frac{q_1}{(1 + q_1)^2}, \\ y_2 &= \frac{q_2}{(1 + q_2)^2}, \\ y_3 &= q_3(1 + q_1)(1 + q_2) \end{aligned}$$

The Batyrev and Seidel elements.

$$\begin{aligned} \tilde{S}_1 &= \tilde{S}_2 = \tilde{D}_1 = \tilde{D}_2 = \tilde{p}_2 = \frac{1 + q_2}{1 - q_2} p_2 - \frac{q_2}{1 - q_2} p_3, \\ \tilde{S}_3 &= \tilde{S}_4 = \tilde{D}_3 = \tilde{D}_4 = \tilde{p}_1 = \frac{1 + q_1}{1 - q_1} p_1 - \frac{q_1}{1 - q_1} p_3, \\ \tilde{S}_5 &= \frac{1}{1 + q_1} \tilde{D}_5, \quad \tilde{D}_5 = -2\tilde{p}_1 + \tilde{p}_3 = \frac{1 + q_1}{1 - q_1} D_5, \\ \tilde{S}_6 &= \frac{1}{1 + q_2} \tilde{D}_6, \quad \tilde{D}_6 = -2\tilde{p}_2 + \tilde{p}_3 = \frac{1 + q_2}{1 - q_2} D_6. \end{aligned}$$

FIGURE 1. Crepant resolution of $(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$

Reconstruction from Seidel elements. The reconstruction in this case becomes a little more involved. Since $\tilde{S}_1, \dots, \tilde{S}_4$ have no correction terms, we have the relations

$$(31) \quad \tilde{S}_1 = \tilde{S}_2 = \sum_{i=1}^3 \frac{\partial \log q_i}{\partial \log y_2} p_i, \quad \tilde{S}_3 = \tilde{S}_4 = \sum_{i=1}^3 \frac{\partial \log q_i}{\partial \log y_1} p_i$$

but these are not enough to determine the mirror coordinates y_1, y_2, y_3 . We also need to require that the mirror map is homogeneous, i.e. the Euler vector field is preserved

$$(32) \quad 2y_3 \frac{\partial}{\partial y_3} = 2q_3 \frac{\partial}{\partial q_3}.$$

The Equations (31), (32) can reconstruct the mirror coordinates y_1, y_2, y_3 .

In general, the method of the reconstruction illustrated here works if $c_1(X)$ together with the divisors D_j for which b_j is a vertex of the fan polytope span $H^2(X)$.

4.3.3. *Crepant resolution of $(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$.* Let \mathbb{Z}_2 act on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$([z_1, z_2], [w_1, w_2]) \mapsto ([z_1, -z_2], [w_1, -w_2]).$$

The quotient $(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$ has four isolated singular points of type A_1 . Let X denote the minimal resolution of $(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$. It is given by the complete regular fan (Figure (1)) spanned by

$$\begin{aligned} b_1 &= (-1, 1), \quad b_2 = (1, 1), \quad b_3 = (1, -1), \quad b_4 = (-1, -1), \\ b_5 &= (0, 1), \quad b_6 = (1, 0), \quad b_7 = (0, -1), \quad b_8 = (-1, 0). \end{aligned}$$

The vertices of the fan polytope are b_1, \dots, b_4 .

Define $\gamma_1, \dots, \gamma_6 \in H_2(X, \mathbb{Z})$ to be the \mathbb{Q} -basis of $H_2(X)$ with the following intersection matrix:

$$(\langle D_i, \gamma_j \rangle)^T = \begin{pmatrix} 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ -2 & -1 & -2 & -1 & 2 & 2 & 2 & 2 \\ -1 & -2 & -1 & -2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

Again each row vector gives a relation of b_1, \dots, b_8 . In this case the Mori cone $\text{NE}(X)$ is not simplicial, but is contained in the cone spanned by $\gamma_1, \dots, \gamma_6$. Let $p_1, \dots, p_6 \in H^2(X)$ be the dual basis of $\gamma_1, \dots, \gamma_6$ and let q_1, \dots, q_6 be the corresponding Novikov variables. Note that $\deg q_1 = \dots = \deg q_4 = 0$ and $\deg q_5 = \deg q_6 = 4$.

Mirror coordinates.

$$y_i = \begin{cases} \frac{q_i}{(1+q_i)^2}, & i = 1, 2, 3, 4 \\ q_i(1+q_1)^2(1+q_2)^2(1+q_3)^2(1+q_4)^2, & i = 5, 6. \end{cases}$$

From this we get the elements

$$\tilde{p}_i = \sum_{k=1}^6 \frac{\partial \log q_k}{\partial \log y_i} p_k = \begin{cases} \frac{1+q_i}{1-q_i} p_i - \frac{2q_i}{1-q_i} p_5 - \frac{2q_i}{1-q_i} p_6, & i = 1, 2, 3, 4. \\ p_i, & i = 5, 6. \end{cases}$$

The Batyrev and Seidel elements.

$$\begin{aligned} \tilde{S}_1 &= \tilde{D}_1 = \tilde{p}_1 + \tilde{p}_4 - 2\tilde{p}_5 - \tilde{p}_6 \\ &= \frac{1+q_1}{1-q_1} p_1 + \frac{1+q_4}{1-q_4} p_4 - \left(\frac{1+q_1}{1-q_1} + \frac{1+q_4}{1-q_4} \right) p_5 - \left(\frac{1+q_1}{1-q_1} + \frac{2q_4}{1-q_4} \right) p_6 \\ \tilde{S}_2 &= \tilde{D}_2 = \tilde{p}_1 + \tilde{p}_2 - \tilde{p}_5 - 2\tilde{p}_6 \\ &= \frac{1+q_1}{1-q_1} p_1 + \frac{1+q_2}{1-q_2} p_2 - \left(\frac{1+q_1}{1-q_1} + \frac{2q_2}{1-q_2} \right) p_5 - \left(\frac{1+q_1}{1-q_1} + \frac{1+q_2}{1-q_2} \right) p_6 \\ \tilde{S}_3 &= \tilde{D}_3 = \tilde{p}_2 + \tilde{p}_3 - 2\tilde{p}_5 - \tilde{p}_6 \\ &= \frac{1+q_2}{1-q_2} p_2 + \frac{1+q_3}{1-q_3} p_3 - \left(\frac{1+q_2}{1-q_2} + \frac{1+q_3}{1-q_3} \right) p_5 - \left(\frac{1+q_2}{1-q_2} + \frac{2q_3}{1-q_3} \right) p_6 \\ \tilde{S}_4 &= \tilde{D}_4 = \tilde{p}_3 + \tilde{p}_4 - \tilde{p}_5 - 2\tilde{p}_6 \\ &= \frac{1+q_3}{1-q_3} p_3 + \frac{1+q_4}{1-q_4} p_4 - \left(\frac{1+q_3}{1-q_3} + \frac{2q_4}{1-q_4} \right) p_5 - \left(\frac{1+q_3}{1-q_3} + \frac{1+q_4}{1-q_4} \right) p_6 \\ \tilde{S}_5 &= \frac{1}{1+q_1} \tilde{D}_5, \quad \tilde{D}_5 = -2\tilde{p}_1 + 2\tilde{p}_5 + 2\tilde{p}_6 = \frac{1+q_1}{1-q_1} (-2p_1 + 2p_5 + 2p_6) = \frac{1+q_1}{1-q_1} D_5 \\ \tilde{S}_6 &= \frac{1}{1+q_2} \tilde{D}_6, \quad \tilde{D}_6 = -2\tilde{p}_2 + 2\tilde{p}_5 + 2\tilde{p}_6 = \frac{1+q_2}{1-q_2} (-2p_2 + 2p_5 + 2p_6) = \frac{1+q_2}{1-q_2} D_6 \\ \tilde{S}_7 &= \frac{1}{1+q_3} \tilde{D}_7, \quad \tilde{D}_7 = -2\tilde{p}_3 + 2\tilde{p}_5 + 2\tilde{p}_6 = \frac{1+q_3}{1-q_3} (-2p_3 + 2p_5 + 2p_6) = \frac{1+q_3}{1-q_3} D_7 \\ \tilde{S}_8 &= \frac{1}{1+q_4} \tilde{D}_8, \quad \tilde{D}_8 = -2\tilde{p}_4 + 2\tilde{p}_5 + 2\tilde{p}_6 = \frac{1+q_4}{1-q_4} (-2p_4 + 2p_5 + 2p_6) = \frac{1+q_4}{1-q_4} D_8. \end{aligned}$$

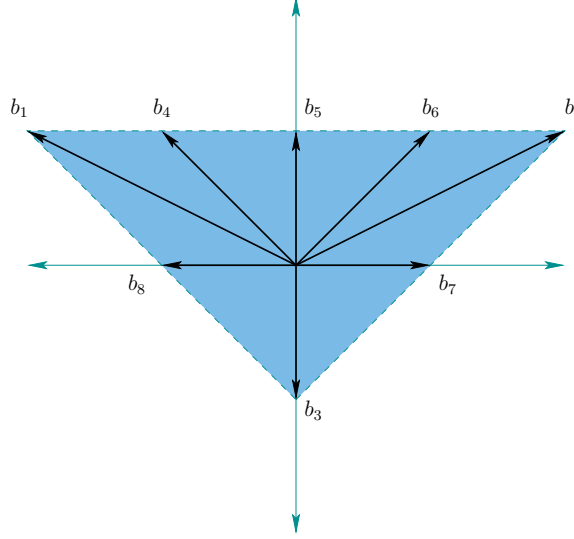


FIGURE 2. Fan polytope of a smooth NEF toric manifold for which the divisors corresponding to vertices and c_1 do not span $H^2(X)$. This gives a crepant resolution of $\mathbb{P}(1, 1, 2)/\mathbb{Z}_2$.

Reconstruction from the Seidel elements. In this case, the divisors D_1, \dots, D_4 and $c_1(X)$ do not span $H^2(X)$ and the method in the previous example does not apply. Assume that we know the Seidel elements $\tilde{S}_1, \dots, \tilde{S}_8$ given above. We will reconstruct the Batyrev elements from these. We set $\tilde{D}_i := \tilde{S}_i$ for $i = 1, 2, 3, 4$ and $\tilde{D}_i := H_i \tilde{S}_i$ for $i = 5, 6, 7, 8$ for some $H_i \in \mathbb{Q}[[q]]$. The linear relation

$$\tilde{D}_1 + \tilde{D}_5 + \tilde{D}_2 = \tilde{D}_4 + \tilde{D}_7 + \tilde{D}_3.$$

gives the equation

$$H_5 \tilde{S}_5 - H_7 \tilde{S}_7 = \tilde{S}_3 + \tilde{S}_4 - \tilde{S}_1 - \tilde{S}_2.$$

From the computation above, we have

$$-\frac{H_7}{1-q_3} D_7 + \frac{H_5}{1-q_1} D_5 = -\frac{1+q_3}{1-q_3} D_7 + \frac{1+q_1}{1-q_1} D_5.$$

Since D_7 and D_5 are linearly independent, it follows that $H_5 = 1+q_1$, $H_7 = 1+q_3$. Similarly one can solve for H_6, H_8 .

4.3.4. *Crepan resolution of $\mathbb{P}(1, 1, 2)/\mathbb{Z}_2$.* Consider the \mathbb{Z}_2 action on the weighted projective space $\mathbb{P}(1, 1, 2)$ given by

$$[z_1, z_2, z_3] \mapsto [-z_1, z_2, -z_3].$$

The quotient $\mathbb{P}(1, 1, 2)/\mathbb{Z}_2$ has three singular points, two of which are type A_1 and the other is of type A_3 . Let X be the minimal resolution of $\mathbb{P}(1, 1, 2)/\mathbb{Z}_2$. The fan of X is given by (see Figure 2)

$$\begin{aligned} b_1 &= (-2, 1), \quad b_2 = (2, 1), \quad b_3 = (0, -1) \\ b_4 &= (-1, 1), \quad b_5 = (0, 1), \quad b_6 = (1, 1), \quad b_7 = (1, 0), \quad b_8 = (-1, 0) \end{aligned}$$

The Picard number of X is 6. The vectors b_1, b_2, b_3 are vertices of the fan polytope.

Let $\{\gamma_1, \dots, \gamma_6\}$ be the basis of $H_2(X, \mathbb{Q})$ such that

$$(\langle D_i, \gamma_j \rangle)^T = \begin{pmatrix} 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ -2 & -2 & -1 & 2 & -1 & 2 & 2 & 2 \end{pmatrix}$$

The Mori cone $\text{NE}(X)$ is not simplicial but is contained in the simplicial cone generated by $\gamma_1, \dots, \gamma_6$. (Here $\gamma_i \in \text{NE}(X)$ for $1 \leq i \leq 5$, but $\gamma_6 \notin \text{NE}(X)$. Note also that $\gamma_1, \dots, \gamma_6$ do not form an integral basis.) Set the Novikov variable $q_i := q^{\gamma_i}$. Let $\{p_1, \dots, p_6\} \subset H^2(X, \mathbb{Q})$ be the dual basis of $\{\gamma_1, \dots, \gamma_6\}$.

Mirror coordinates. The mirror coordinates for surface A_n singularity resolutions are calculated in [3, Appendix A, Proposition A.6]. The same method applies here, as the mirror map for the A_3 singularity resolution appears as part of the mirror map in this case. We only state the final result here.

$$\begin{aligned} y_1 &= q_1 \frac{1 + q_2 + q_1 q_2 + q_2 q_3 + q_1 q_2 q_3 + q_1 q_2^2 q_3}{(1 + q_1 + q_1 q_2 + q_1 q_2 q_3)^2}, \\ y_2 &= q_2 \frac{(1 + q_1 + q_1 q_2 + q_1 q_2 q_3)(1 + q_3 + q_2 q_3 + q_1 q_2 q_3)}{(1 + q_2 + q_1 q_2 + q_2 q_3 + q_1 q_2 q_3 + q_1 q_2^2 q_3)^2}, \\ y_3 &= q_3 \frac{1 + q_2 + q_1 q_2 + q_2 q_3 + q_1 q_2 q_3 + q_1 q_2^2 q_3}{(1 + q_3 + q_2 q_3 + q_1 q_2 q_3)^2}, \\ y_4 &= \frac{q_4}{(1 + q_4)^2}, \\ y_5 &= \frac{q_5}{(1 + q_5)^2}, \\ y_6 &= q_6 \frac{(1 + q_1 + q_1 q_2 + q_1 q_2 q_3)^2 (1 + q_3 + q_2 q_3 + q_1 q_2 q_3)^2 (1 + q_4)^2 (1 + q_5)^2}{1 + q_2 + q_1 q_2 + q_2 q_3 + q_1 q_2 q_3 + q_1 q_2^2 q_3}. \end{aligned}$$

The Batyrev and Seidel elements. Here we present a cohomology class $\sum_{i=1}^6 c_i p_i$ by the column vector $(c_1, c_2, \dots, c_6)^T$.

$$\tilde{D}_1 = \tilde{S}_1 = \begin{pmatrix} \frac{1+q_1-q_2-q_2q_3-q_1^2q_2+q_2^2q_3-q_1^2q_2q_3+q_1^3q_2^2q_3}{(1-q_1)(1-q_2)(1-q_1q_2)(1-q_2q_3)(1-q_1q_2q_3)} \\ -\frac{q_1(1-q_3-q_1q_2+q_2^2-q_1q_2^2+q_1q_2q_3-q_2^3q_3+q_1q_2^3q_3)}{(1-q_1)(1-q_2)(1-q_3)(1-q_1q_2)(1-q_2q_3)} \\ \frac{q_1q_2^2(1-q_2q_3-q_1q_2q_3+q_3^3-q_2q_3^3+q_1q_2^2q_3^2-q_1q_2q_3^3+q_1q_2^2q_3^3)}{(1-q_2)(1-q_3)(1-q_1q_2)(1-q_2q_3)(1-q_1q_2q_3)} \\ 0 \\ \frac{1+q_5}{1-q_5} \\ \frac{1-3q_5}{1-q_5} + \frac{q_1}{(1-q_1)(1-q_2)(1-q_2q_3)} - \frac{3}{(1-q_1)(1-q_1q_2)(1-q_1q_2q_3)} - \frac{2q_1q_2^2}{(1-q_2)(1-q_3)(1-q_1q_2)} \end{pmatrix}$$

$$\tilde{D}_2 = \tilde{S}_2 = \begin{pmatrix} \frac{q_2^2 q_3 (1-q_1 q_2 + q_1^3 - q_1 q_2 q_3 - q_1^3 q_2 - q_1^3 q_2 q_3 + q_1^2 q_2^2 q_3 + q_1^3 q_2^2 q_3)}{(1-q_1)(1-q_2)(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ - \frac{q_3 (1-q_1 + q_2^2 - q_2 q_3 + q_1 q_2 q_3 - q_2^2 q_3 - q_1 q_2^3 + q_1 q_2^3 q_3)}{(1-q_1)(1-q_2)(1-q_3)(1-q_1 q_2)(1-q_2 q_3)} \\ \frac{1-q_2+q_3-q_1 q_2+q_1 q_2^2-q_2 q_3^2-q_1 q_2 q_3^2+q_1 q_2^2 q_3^3}{(1-q_2)(1-q_3)(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ \frac{1+q_4}{1-q_4} \\ 0 \\ \frac{1-3q_4}{1-q_4} + \frac{q_3}{(1-q_2)(1-q_3)(1-q_1 q_2)} - \frac{3}{(1-q_3)(1-q_2 q_3)(1-q_1 q_2 q_3)} - \frac{2q_2^2 q_3}{(1-q_1)(1-q_2)(1-q_2 q_3)} \end{pmatrix}$$

$$\tilde{D}_3 = \tilde{S}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1+q_4}{1-q_4} \\ \frac{1+q_5}{1-q_5} \\ -\frac{2}{1-q_4} + \frac{1-3q_5}{1-q_5} \end{pmatrix}$$

$$\tilde{D}_4 = (1 + q_1 + q_1 q_2 + q_1 q_2 q_3) \tilde{S}_4,$$

$$\tilde{S}_4 = \begin{pmatrix} -\frac{2-q_2-q_1 q_2-q_2 q_3-q_1 q_2 q_3+q_2^3 q_3+q_1^2 q_2^2 q_3}{(1-q_1)(1-q_2)(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ \frac{1+q_2-q_3-2q_1 q_2+q_1 q_2 q_3-q_2^3 q_3+q_1 q_2^2 q_3}{(1-q_1)(1-q_2)(1-q_3)(1-q_1 q_2)(1-q_2 q_3)} \\ -\frac{q_2(1-q_2 q_3+q_2^2-q_1 q_2 q_3-q_2 q_3^2-q_1 q_2 q_3^2+2q_1 q_2^2 q_3^2)}{(1-q_2)(1-q_3)(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ 0 \\ 0 \\ -\frac{1}{(1-q_1)(1-q_2)(1-q_2 q_3)} + \frac{3}{(1-q_1)(1-q_1 q_2)(1-q_1 q_2 q_3)} + \frac{2q_2}{(1-q_2)(1-q_3)(1-q_1 q_2)} \end{pmatrix}$$

$$\tilde{D}_5 = (1 + q_2 + q_1 q_2 + q_2 q_3 + q_1 q_2 q_3 + q_1 q_2^2 q_3) \tilde{S}_5$$

$$\tilde{S}_5 = \begin{pmatrix} \frac{1+q_1-2q_1 q_2-2q_1 q_2 q_3+q_1 q_2^2 q_3+q_1^2 q_2^2 q_3}{(1-q_1)(1-q_2)(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ -\frac{2-q_1-q_3-q_1 q_2-q_2 q_3+2q_1 q_2 q_3}{(1-q_1)(1-q_2)(1-q_3)(1-q_1 q_2)(1-q_2 q_3)} \\ \frac{1+q_3-2q_2 q_3-2q_1 q_2 q_3+q_1 q_2^2 q_3+q_1 q_2^2 q_3^2}{(1-q_2)(1-q_3)(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ 0 \\ 0 \\ \frac{1}{(1-q_1)(1-q_2)(1-q_2 q_3)} - \frac{2}{(1-q_2)(1-q_3)(1-q_1 q_2)} - \frac{3q_1}{(1-q_1)(1-q_1 q_2)(1-q_1 q_2 q_3)} \end{pmatrix}$$

$$\tilde{D}_6 = (1 + q_3 + q_2 q_3 + q_1 q_2 q_3) \tilde{S}_6$$

$$\tilde{S}_6 = \begin{pmatrix} -\frac{q_2(1+q_1^2-q_1 q_2-q_1^2 q_2-q_1 q_2 q_3-q_1^2 q_2 q_3+2q_1^2 q_2^2 q_3)}{(1-q_1)(1-q_2)(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ \frac{1-q_1+q_2-2q_2 q_3-q_1 q_2^2+q_1 q_2 q_3+q_1 q_2^2 q_3}{(1-q_1)(1-q_2)(1-q_3)(1-q_1 q_2)(1-q_2 q_3)} \\ -\frac{2-q_2-q_1 q_2-q_2 q_3+q_1 q_2^2-q_1 q_2 q_3+q_1 q_2^2 q_3^2}{(1-q_2)(1-q_3)(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ 0 \\ 0 \\ -\frac{1}{(1-q_2)(1-q_3)(1-q_1 q_2)} + \frac{3}{(1-q_3)(1-q_2 q_3)(1-q_1 q_2 q_3)} + \frac{2q_2}{(1-q_1)(1-q_2)(1-q_2 q_3)} \end{pmatrix}$$

$$\tilde{D}_7 = (1 + q_4)\tilde{S}_7, \quad \tilde{S}_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{2}{1-q_4} \\ 0 \\ \frac{2}{1-q_4} \end{pmatrix}, \quad \tilde{D}_8 = (1 + q_5)\tilde{S}_8, \quad \tilde{S}_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{2}{1-q_5} \\ \frac{2}{1-q_5} \end{pmatrix}.$$

Reconstruction of Batyrev from Seidel. Again the divisors D_1, D_2, D_3 and $c_1(X)$ do not span $H^2(X)$ in this case. We have the following linear relations:

$$\begin{aligned} 2\tilde{D}_1 + \tilde{D}_4 + \tilde{D}_8 &= \tilde{D}_6 + \tilde{D}_7 + 2\tilde{D}_2, \\ \tilde{D}_1 + \tilde{D}_4 + \tilde{D}_5 + \tilde{D}_6 + \tilde{D}_2 &= \tilde{D}_3. \end{aligned}$$

Suppose we know only the Seidel elements $\tilde{S}_1, \dots, \tilde{S}_8$ given above. We can check (assisted by computer) that there exist unique functions x, y, z, u, v of q_i 's which solves the linear equations:

$$\begin{aligned} 2\tilde{S}_1 + x\tilde{S}_4 + v\tilde{S}_8 &= z\tilde{S}_6 + u\tilde{S}_7 + 2\tilde{S}_2, \\ \tilde{S}_1 + x\tilde{S}_4 + y\tilde{S}_5 + z\tilde{S}_6 + \tilde{S}_2 &= \tilde{S}_3. \end{aligned}$$

Here x, y, z, u, v are given by (as expected)

$$\begin{aligned} x &= 1 + q_1 + q_1q_2 + q_1q_2q_3 \\ y &= 1 + q_2 + q_1q_2 + q_2q_3 + q_1q_2q_3 + q_1q_2^2q_3 \\ z &= 1 + q_3 + q_2q_3 + q_1q_2q_3 \\ u &= 1 + q_4 \\ v &= 1 + q_5. \end{aligned}$$

The Batyrev elements are given as: $\tilde{D}_i = \tilde{S}_i$ for $1 \leq i \leq 3$ and $\tilde{D}_4 = x\tilde{S}_4$, $\tilde{D}_5 = y\tilde{S}_5$, $\tilde{D}_6 = z\tilde{S}_6$, $\tilde{D}_7 = u\tilde{S}_7$, $\tilde{D}_8 = v\tilde{S}_8$. Then the Batyrev elements determine the mirror co-ordinates y_1, \dots, y_6 .

4.3.5. Crepant resolution of $\mathbb{P}^2/\mathbb{Z}_3$. Consider the toric variety X given by the fan spanned by the following vectors (see Figure 3):

$$\begin{aligned} b_1 &= (0, 1), \quad b_2 = (1, 0), \quad b_3 = (1, -1), \quad b_4 = (0, -1), \quad b_5 = (-1, 0), \quad b_6 = (-1, 1) \\ b_7 &= (-1, 2), \quad b_8 = (2, -1), \quad b_9 = (-1, -1) \end{aligned}$$

Only b_7, b_8, b_9 are the vertices of the fan polytope. A cubic surface in \mathbb{P}^3 can degenerate to the singular toric variety $\mathbb{P}^2/\mathbb{Z}_3$ (with 3 nodes of type A_2) and X is a minimal resolution of $\mathbb{P}^2/\mathbb{Z}_3$ where the \mathbb{Z}_3 -action on \mathbb{P}^2 is given by $[z_1, z_2, z_3] \mapsto [z_1, \omega z_2, \omega^2 z_3]$. We can construct X as a 6 times blowup of \mathbb{P}^2 at infinitely near points. Thus X is deformation equivalent to a del Pezzo surface of degree 3 (i.e. cubic surface).

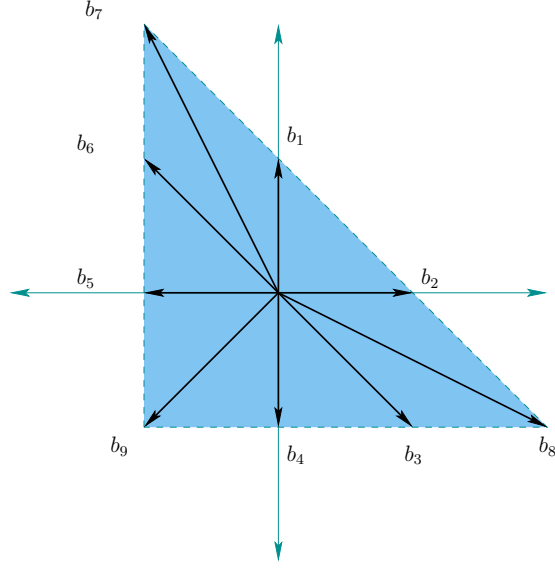


FIGURE 3. Toric degeneration of a cubic surface

We take a basis $\{\gamma_1, \dots, \gamma_7\}$ of $H_2(X, \mathbb{Q})$ such that the intersection matrix becomes

$$(\langle D_i, \gamma_j \rangle)^T = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & -3 & -3 & -3 \end{pmatrix}$$

The Mori cone $\text{NE}(X)$ is contained in the simplicial cone generated by γ_i 's. Note that $\gamma_1, \dots, \gamma_7$ are not a \mathbb{Z} -basis and that $\text{NE}(X)$ itself is not simplicial. We have $\gamma_i \in \text{NE}(X)$, $1 \leq i \leq 6$ and $\gamma_7 \notin \text{NE}(X)$. As usual, we take $\{p_1, \dots, p_7\} \subset H^2(X, \mathbb{Q})$ to be the dual basis of $\{\gamma_1, \dots, \gamma_7\}$ and $q_i := q^{\gamma_i}$ to be the Novikov variable.

Mirror coordinates.

$$\begin{aligned}
y_1 &= q_1 \frac{1 + q_2 + q_1 q_2}{(1 + q_1 + q_1 q_2)^2} \\
y_2 &= q_2 \frac{1 + q_1 + q_1 q_2}{(1 + q_2 + q_1 q_2)^2} \\
y_3 &= q_3 \frac{1 + q_4 + q_3 q_4}{(1 + q_3 + q_3 q_4)^2} \\
y_4 &= q_4 \frac{1 + q_3 + q_3 q_4}{(1 + q_4 + q_3 q_4)^2} \\
y_5 &= q_5 \frac{1 + q_6 + q_5 q_6}{(1 + q_5 + q_5 q_6)^2} \\
y_6 &= q_6 \frac{1 + q_5 + q_5 q_6}{(1 + q_6 + q_5 q_6)^2} \\
y_7 &= q_7 (1 + q_1 + q_1 q_2)^2 (1 + q_2 + q_1 q_2)^2 (1 + q_3 + q_3 q_4)^2 \\
&\quad \cdot (1 + q_4 + q_3 q_4)^2 (1 + q_5 + q_5 q_6)^2 (1 + q_6 + q_5 q_6)^2
\end{aligned}$$

The Batyrev and Seidel elements.

$$\begin{aligned}
\tilde{D}_1 &= (1 + q_1 + q_1 q_2) \tilde{S}_1, \quad \tilde{S}_1 = \frac{1}{\Delta_{12}} \begin{pmatrix} -2 + q_2 + q_1 q_2 \\ 1 + q_2 - 2q_1 q_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2(1 - 2q_2 + q_1 q_2) \end{pmatrix} \\
\tilde{D}_2 &= (1 + q_2 + q_1 q_2) \tilde{S}_2, \quad \tilde{S}_2 = \frac{1}{\Delta_{12}} \begin{pmatrix} 1 + q_1 - 2q_1 q_2 \\ -2 + q_1 + q_1 q_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2(1 - 2q_1 + q_1 q_2) \end{pmatrix} \\
\tilde{D}_3 &= (1 + q_3 + q_3 q_4) \tilde{S}_3, \quad \tilde{S}_3 = \frac{1}{\Delta_{34}} \begin{pmatrix} 0 \\ 0 \\ -2 + q_4 + q_3 q_4 \\ 1 + q_4 - 2q_3 q_4 \\ 0 \\ 0 \\ 2(1 - 2q_4 + q_3 q_4) \end{pmatrix} \\
\tilde{D}_4 &= (1 + q_4 + q_3 q_4) \tilde{S}_4, \quad \tilde{S}_4 = \frac{1}{\Delta_{34}} \begin{pmatrix} 0 \\ 0 \\ 1 + q_3 - 2q_3 q_4 \\ -2 + q_3 + q_3 q_4 \\ 0 \\ 0 \\ 2(1 - 2q_3 + q_3 q_4) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\tilde{D}_5 &= (1 + q_5 + q_5 q_6) \tilde{S}_5, & \tilde{S}_5 &= \frac{1}{\Delta_{56}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 + q_6 + q_5 q_6 \\ 1 + q_6 - 2q_5 q_6 \\ 2(1 - 2q_6 + q_5 q_6) \end{pmatrix} \\
\tilde{D}_6 &= (1 + q_6 + q_5 q_6) \tilde{S}_6, & \tilde{S}_6 &= \frac{1}{\Delta_{56}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 + q_5 - 2q_5 q_6 \\ -2 + q_5 + q_5 q_6 \\ 2(1 - 2q_5 + q_5 q_6) \end{pmatrix} \\
\tilde{D}_7 &= \tilde{S}_7 = \frac{1}{\Delta_{12} \Delta_{56}} \begin{pmatrix} (1 + q_1 - q_2 - q_1^2 q_2) \Delta_{56} \\ -q_1(1 - q_1 q_2 + q_2^2 - q_1 q_2^2) \Delta_{56} \\ 0 \\ 0 \\ -q_6(1 + q_5^2 - q_5 q_6 - q_5^2 q_6) \Delta_{12} \\ (1 - q_5 + q_6 - q_5 q_6^2) \Delta_{12} \\ 5 - 2(2 - q_1 - 2q_2 + q_1^2 q_2) \Delta_{12}^{-1} - 2(2 - 2q_5 - q_6 + q_5 q_6^2) \Delta_{56}^{-1} \end{pmatrix} \\
\tilde{D}_8 &= \tilde{S}_8 = \frac{1}{\Delta_{12} \Delta_{34}} \begin{pmatrix} -q_2(1 - q_1 q_2 + q_1^2 - q_1^2 q_2) \Delta_{34} \\ (1 - q_1 + q_2 - q_1 q_2^2) \Delta_{34} \\ (1 + q_3 - q_4 - q_3^2 q_4) \Delta_{12} \\ -q_3(1 - q_3 q_4 + q_4^2 - q_3 q_4^2) \Delta_{12} \\ 0 \\ 0 \\ 5 - 2(2 - 2q_1 - q_2 + q_1 q_2^2) \Delta_{12}^{-1} - 2(2 - q_3 - 2q_4 + q_3^2 q_4) \Delta_{34}^{-1} \end{pmatrix} \\
\tilde{D}_9 &= \tilde{S}_9 = \frac{1}{\Delta_{34} \Delta_{56}} \begin{pmatrix} 0 \\ 0 \\ -q_4(1 + q_3^2 - q_3 q_4 - q_3^2 q_4) \Delta_{56} \\ (1 - q_3 + q_4 - q_3 q_4^2) \Delta_{56} \\ (1 + q_5 - q_6 - q_5^2 q_6) \Delta_{34} \\ -q_5(1 - q_5 q_6 - q_5 q_6^2 + q_6^2) \Delta_{34} \\ 5 - 2(2 - 2q_3 - q_4 + q_3 q_4^2) \Delta_{34}^{-1} - 2(2 - q_5 - 2q_6 + q_5^2 q_6) \Delta_{56}^{-1} \end{pmatrix}
\end{aligned}$$

where $\Delta_{ij} = (1 - q_i)(1 - q_j)(1 - q_i q_j)$.

Reconstruction of Batyrev from Seidel. We have the following linear relations:

$$\begin{aligned}
2\tilde{D}_7 + \tilde{D}_1 + \tilde{D}_6 &= \tilde{D}_9 + \tilde{D}_4 + \tilde{D}_3 + \tilde{D}_8 \\
2\tilde{D}_8 + \tilde{D}_2 + \tilde{D}_3 &= \tilde{D}_9 + \tilde{D}_5 + \tilde{D}_6 + \tilde{D}_7
\end{aligned}$$

Suppose we only know the Seidel elements $\tilde{S}_1, \dots, \tilde{S}_9$. We can check that the following linear equation for x, y, z, w, u, v has a unique solution:

$$\begin{aligned}
2\tilde{S}_7 + x\tilde{S}_1 + v\tilde{S}_6 &= \tilde{S}_9 + w\tilde{S}_4 + z\tilde{S}_3 + \tilde{S}_8 \\
2\tilde{S}_8 + y\tilde{S}_2 + z\tilde{S}_3 &= \tilde{S}_9 + u\tilde{S}_5 + v\tilde{S}_6 + \tilde{S}_7
\end{aligned}$$

The Batyrev elements are given as $\tilde{D}_1 = x\tilde{S}_1$, $\tilde{D}_2 = y\tilde{S}_2$, $\tilde{D}_3 = z\tilde{S}_3$, $\tilde{D}_4 = w\tilde{S}_4$, $\tilde{D}_5 = u\tilde{S}_5$, $\tilde{D}_6 = v\tilde{S}_6$ and $\tilde{D}_i = \tilde{S}_i$ for $7 \leq i \leq 9$. The mirror co-ordinates y_i are determined by these.

Remark 4.7. It is interesting to compare the Givental-Hori-Vafa mirrors of X and a cubic surface. The mirror of X is a Landau-Ginzburg model defined by the function W_y on the torus $(\mathbb{C}^\times)^2$ with coordinates x_1, x_2

$$W_y(x_1, x_2) = a_7 x_1^{-1} x_2^2 + x_2 + x_1 + a_8 x_1^2 x_2^{-1} \\ + a_3 x_1 x_2^{-1} + a_4 x_2^{-1} + a_9 x_1^{-1} x_2^{-1} + a_5 x_1^{-1} + a_6 x_1^{-1} x_2$$

where the coefficients a_3, \dots, a_9 are determined by the relation

$$y_j = \prod_{i=3}^9 a_i^{\langle D_i, \gamma_j \rangle}, \quad 1 \leq j \leq 7.$$

On the other hand, the mirror of a (generic) cubic surface Y is [18]

$$V_u(x_1, x_2) = u \frac{(1 + x_1 + x_2)^3}{x_1 x_2}.$$

Under the specialization $y_1 = \dots = y_6 = 1/3$ and $y_7 = 3^{12} u^3$, we have

$$V_u(x_1, x_2) = W_y(3ux_1, 3ux_2) + 6u.$$

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